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1 Definitions

Borel-Weil constructred a theorem that irreducible representations are the global sections of certain line bundles. Bott calculated the sheaf cohomology of all relevant line bundles. This theorem 'geometrizes' representation theory by interpreting highest weight representations as cohomology groups of line bundles on flag varieties. These global sections of a suitably chosen line bundle give an irreducible representation.
Definition. Cohomology arises from a cochain complex in an abelian category. A cochain complex (C, d) is a sequence of abelian groups and homomorphisms

$$\cdots \to^{d^{n-2}} C^{n-1} \to^{d^{n-1}} C^{n} \to^{d^{n}} C^{n+1} \to^{d^{n+1}} \cdots$$

For each integer n, the differential maps

$$d^n \circ d^{n-1} = 0$$

The nth cohomology group H^n is defined as the quotient

$$H^n(C) = \frac{\ker(d^n : C^n \to C^{n+1})}{\operatorname{im}(d^{n-1} : C^{n-1} \to C^n)}$$

We therefore look at this quotient to see the 'obstructions' to writing a cocycle as a coboundary.
The most 'general' form of Borel-Weil-Bott is the Beilinson-Bernstein theorem. This theorem sets up a correspondence between all representations of a semisimple Lie algebra (not just finite dimensional irreducible representations) and certain geometric objects called D-modules.

Definition. A Lie algebra $\mathfrak{g}$ is semi-simple if all its solvable ideals are 0.

Cartan Criterion. A Lie algebra $\mathfrak{g}$ is semi-simple if and only the Killing form $Tr(ad_x ad_y)$ is non degenerate. Recall the Killing form from earlier in the course. It's a symmetric bilinear form:

$$B(x, y) = \operatorname{Tr}(\operatorname{ad}(x)\operatorname{ad}(y))$$

Non-degenerate if whenever $B(x, y) = 0$ for all $y \in \mathfrak{g}$, it follows that $x = 0$.

Weyl's Complete Reducibility Theorem. Every finite dimensional representation of a semi-simple Lie algebra splits as the direct sum of irreducible representations, which I talked about in previous talks.

Definition. A subalgebra $\mathfrak{h}$ of a semisimple Lie algebra $\mathfrak{g}$ is called a Cartan subalgebra if it is a maximal abelian subalgebra having the property that ad_h is diagonizable for all $h \in \mathfrak{h}$

Definition. A Flag, for a finite-dimensional vector space V, is a chain of nested subspaces:

$$\{0\} \subset V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_k = V$$

The flag varieties come from $SL(n, \mathbb{C})$. We can associate a complete flag to any given element $g \in SL(n, \mathbb{C})$. If the the columns of g are the vectors $v_1, \ldots, v_n \in \mathbb{C}^n$, we can create these nested subspaces

$$\{0\} \subset \operatorname{Span}(v_1) \subset \operatorname{Span}(v_1, v_2) \subset \cdots \subset \operatorname{Span}(v_1, \ldots, v_n) = \mathbb{C}^n$$

Example:

$$\mathbb{C}^3, g \in GL(3)$$

$$g = \begin{pmatrix} g_{11} & g_{12} & g_{12} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix}$$

$$0 \subset g \cdot \mathrm{span}(e_1) = \mathrm{span}(ge_1) \subset g \cdot \mathrm{span}(e_1, e_2) = \mathrm{span}(ge_1, ge_2) \subset g \cdot \mathrm{span}(e_1, e_2, e_3) = \mathbb{C}^3$$

Not every g produces a distinct flag. Multiplying by an upper-triangular matrix $b \in B$ preserves the flag structure. The subspace spanned by the first columns is unchanged after rescaling and linear combinations that occur. A flag variety of a semi-simple algebraic group G is the quotient G/B. We will call this flag variety X from here on out.

Definition. A vector bundle consists of three pieces; the total space V, a base space X, and the continuous projection map $\pi : V \to X$ which sends each vector in the total space to the point in X over which it lies. Example: the trivial bundle $E \cong X \times V$; we take the projection map $\pi(x, v) = x$.

Definition. Given a line bundle V, the **fiber** over a point $x \in X$ is defined as the set of all points in V that map to x under our projection π:

$$V_x = \pi^{-1}(x)$$

Definition. Take a topological space X with a continuous $G-$action. A vector bundle $\pi : V \to X$ is called G-equivariant if V also has a continuous G action such that $g \in G$ sends the fiber V_x to V_{gx}, and the map $g\cdot : V_x \to V_{gx}$ is a linear isomorphism. We can have the following commutation with this diagram:

$$\begin{tikzcd} \mathrm{V} \arrow[r, "g\cdot"'] & \mathrm{V} \\ \mathrm{X} \arrow[u, "s"] \arrow[r, "g\cdot"] & \mathrm{X} \arrow[u, "s"'] \end{tikzcd}$$

Define that $(g \cdot s)(x) = gs(g^{-1}x)$.

2 Borel-Weil-Bott

Let G be a simply-connected semisimple algebraic group over $\mathbb{C}$ with Lie algebra $\mathfrak{g}$. Because of our assumption that G is simply connected, we have Lie group-Lie algebra correspondence. The correspondence between subgroups of G and subalgebras of $\mathfrak{g}$ is as folllows: H to $\mathfrak{h}$ is called a maximal torus of G, B to $\mathfrak{b}$ is a Borel subgroup containing H, and $N \subset B$ to $\mathfrak{n}$ has the property that the quotient $B/N \cong H$.

We have the identity matrix, which gives a pretty easy flag whose stabilizer is the group of upper triangular matrices B. SL_n acts on flags by transforming each of the subspaces as usual. It acts transitively on complete flags, and the set of complete flags in $\mathbb{C}^n$ has a bijection with SL_2/B, where B is a Borel subgroup. Any quotient G/B of an algebraic group by a Borel subgroup is called a flag variety.

Proposition. G-equivariant vector bundles on the flag variety X are in 1-1 correspondence with representations of B.

Take a representation V of B, with action bv, and construct a G-equivariant vector bundle as follows. Take the trivial bundle $G \times V$, with the action of B given by

$$b \cdot (g, v) = (gb^{-1}, bv)$$

Take the quotient of this action, get the space $G \times_B V$:

$$\pi : G \times_B V \to X$$

This is built from our projection to the first factor, and form the equivalence relation by

$$(gb, v) \sim (g, b \cdot v)$$

Let us take a B representation V, with a G-equivariant vector bundle E with this correspondence:

$$V \to G \times_B V$$

$$E \to E_{eB}$$

Take our projection $\pi : E \to X$. We can then recover the representation of B on the fiber E_{eB}. This is possible since the fiber is invariant under the action of B, with the stabilizer eB.

Long story short, we have an isomorphism $(G \times_B V)_{eB} \cong V$.

We want to show that this correspondence is 1-1. Let us take our equivariant vector bundle $\mathfrak{V}$ over G/B. Take V to be the fiber $\mathfrak{V}_{eB}$ (thought of as a B-representation). We can find an isomorphism of bundles by defining our function $\varphi : G \times_B V \to \mathfrak{V}$ by

$$\varphi(g, v) = g \cdot v$$

Definition. Let us take the set of positive roots of G determined by our choice of B by Δ^+. Recall that roots are nonzero weights with respect to the maximal torus T where $\mathfrak{g}_\alpha \neq 0$. Take ρ to be half of the sum of the positive roots:

$$\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$$

Definition. We know that we can decompose $V = \oplus V_\lambda$ for weights λ. We know that it will have a unique maximal weight λ and a unique minimal weight (denoted by μ).

Definition. Let λ be an integral weight (a character of T), and let $\mathfrak{L}_\lambda$ denote the holomorphic line bundle on the flag variety X associated to λ.

Formal definition of the Theory. There are two cases:

1. If $\lambda + p$ is **singular**, i.e. there exists a positive root α where the coroot is such that $<\lambda + \rho, \alpha^V> = 0$. Another way to define this is by creating new sets of weights:

$$P = \{\lambda \in \mathfrak{h} | \alpha_i^V(\lambda) \in \mathbb{Z}\}$$

$$P_{sing} = \{\lambda \in P | \alpha^V(\lambda - p) = 0, \alpha \in \Delta^+\}$$

   In this case, we find that we have no cohomology:

$$H^i(X, \mathfrak{L}(\lambda)) = 0 \;\; \forall i \geq 0$$

2. If $\lambda \notin P_{sing}$, then we have a unique element in our Weyl group $w \in W$ such that $w(\lambda + \rho)$ is **dominant**: namely, it lies in the closure of the positive Weyl chamber. We can also define the shifted action of W by a new formula, $w \star \lambda = w(\lambda - \rho) + \rho$. Thus, if $w(\lambda + \rho)$ is dominant, then $w \star \lambda$ is anti-dominant. We have the following structure:

$$H^i(X, \mathfrak{L}_\lambda) \cong \begin{cases} 0 & i \neq l(w) \\ L^{-1}(w \star \lambda) & i = l(w) \end{cases}$$

   We have $L^{-1}(w \star \lambda)$ as a unique representation for having the lowest weight (hence anti-dominant).

If we look at vector bundles on a flag variety, a space which encodes nested subspaces, we see the irreducible representations of a lie gropu G. Depending on the weight λ, and adjusted by a particular shift ρ, all cohomology groups either vanish or exactly one of them is nonzero and it gives you a very specific irreducible representation (of the anti-dominant weight $w \star \lambda$).

Example: The Case of $G = \mathrm{SL}_2(\mathbb{C})$ and Line Bundles on $\mathbb{P}^1$

To make these abstract ideas more tangible, we turn to the familiar example where $G = \mathrm{SL}_2(\mathbb{C})$. In this case, the flag variety is simply the set of all lines through the origin in $\mathbb{C}^2$, which can naturally be identified with the projective line $\mathbb{P}^1$.
The Borel subgroup B that stabilizes the "x-axis" consists of matrices of the form:

$$B = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \;\middle|\; a \in \mathbb{C}^\times,\ b \in \mathbb{C} \right\}.$$

Using homogeneous coordinates $[a : b]$ on $\mathbb{P}^1$, the projection from G to $\mathbb{P}^1$ is given by sending a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \to \quad [a : c].$$

This makes sense because the first column of the matrix captures the line we are interested in when constructing the flag variety.
The Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ has our standard basis:

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

with commutation relations:

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

From these relations, we see that the adjoint action of h diagonalizes nicely: $\mathrm{ad}(h)$ acts with eigenvalues 2, -2, and 0 on e, f, and h respectively.
If we define the character $\rho : \mathfrak{h} \to \mathbb{C}$ by

$$\rho(ah) = a,$$

then the root system is neatly described by:

$$\Delta = \{\pm 2\rho\}, \quad \Delta^+ = \Pi = \{2\rho\}, \quad \pi = \rho, \quad W = \{\pm 1\}.$$

The weight lattice is $P = \mathbb{Z}\rho$, and the positive root lattice is $Q^+ = \mathbb{N}2\rho$.
Choosing 2ρ as the positive root gives a Borel subalgebra spanned by h and e, which corresponds to the Lie algebra of the Borel subgroup B.

We now turn to the Borel-Weil-Bott construction for line bundles over $\mathbb{P}^1$. Given any integer $n \in \mathbb{Z}$, we associate to the weight $n\rho$ a line bundle $L(n\rho)$.
Let $[u : v]$ be homogeneous coordinates on $\mathbb{P}^1$. We cover $\mathbb{P}^1$ by two standard affine charts:

- U_u: where $u \neq 0$, with coordinate $z = v/u$,
- U_v: where $v \neq 0$, with coordinate $x = u/v$.

On the overlap $U_u \cap U_v$, the coordinates relate by $x = z^{-1}$.
Generic elements in these charts correspond, respectively, to the matrices:

$$\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \quad (\text{on } U_u), \qquad \begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix} \quad (\text{on } U_v).$$

An important point here: no nontrivial element of B fixes these matrices, meaning that once we fix the first matrix, the second entry (an element $t \in \mathbb{C}$) is uniquely determined. Thus, the bundle is trivial over both U_u and U_v.
To understand the transition function, we compute how a matrix in U_u moves into the chart U_v. Under the equivalence relation defining the bundle:

$$\left[\left(\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}, t\right)\right] \sim \left[\left(\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}\begin{pmatrix} z^{-1} & -1 \\ 0 & z \end{pmatrix}, z^n t\right)\right] = \left[\left(\begin{pmatrix} z^{-1} & -1 \\ 1 & 0 \end{pmatrix}, z^n t\right)\right].$$

Thus, the transition function multiplies by z^n, matching the transition functions for the sheaf $\mathcal{O}_{\mathbb{P}^1}(-n)$. We conclude:

$$L(n\rho) \cong \mathcal{O}_{\mathbb{P}^1}(-n).$$

Next, we describe the action of G on these bundles. Recall: G acts on sections by

$$g \cdot (g'B, v) = (gg'B, v).$$

Working over U_v, this action becomes:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (x, v) = \left(\frac{ax+b}{cx+d}, (cx+d)^n v \right).$$

This formula should feel familiar to anyone who has encountered modular forms before.
For $n \geq 0$, the bundle $L(-n\rho) = \mathcal{O}(n)$ admits global sections. Over U_v, the section corresponding to the monomial $u^k v^{n-k}$ restricts to x^k. Thus, the space of global sections is:

$$\bigoplus_{k=0}^{n} \mathbb{C}x^k.$$

If $s(x) = x^k$, we can compute how g acts by first evaluating:

$$g^{-1} \cdot x = \frac{dx - b}{-cx + a},$$

then pulling back the section:

$$s(g^{-1}x) = \left(\frac{dx-b}{-cx+a} \right)^k,$$

and finally adjusting by the factor $(cx+d)^n$:

$$g \cdot s = (cx+d)^n \left(\frac{dx-b}{-cx+a} \right)^k.$$

Simplifying yields:

$$g \cdot u^k v^{n-k} = (du - bv)^k (-cu + av)^{n-k}.$$

Finally, to understand the cohomology of these line bundles via Borel-Weil-Bott, observe the following:
The coroot $(2\rho)^\vee$ acts as multiplication by $1/2$, and the shifted action of -1 in the Weyl group is:

$$-1 \star n\rho = (2-n)\rho.$$

Thus, for $n \leq 0$, $n\rho$ lies in the dominant chamber, and we get:

$$L_-(n\rho) = H^0(\mathbb{P}^1, \mathcal{O}(-n)),$$

realizing the standard irreducible representations.
For $n \geq 2$, we must apply Serre duality and the shifted action, giving:

$$H^1(\mathbb{P}^1, L(n\rho)) \cong H^0(\mathbb{P}^1, L((2-n)\rho)) = L_-((2-n)\rho) = L_-(-1 \star n\rho).$$